JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 20, No. 4, July–August 1997

Stochastic Analysis of the Interception of Maneuvering Antisurface Missiles

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The interception of a maneuvering antisurface missile, as in ballistic missile defense and ship defense scenarios, is formulated as an imperfect information, zero-sum, pursuit-evasion game with a state constraint imposed on the evader. Assuming that the perfect information version of the game does not yield a successful result for the defense, the solution of this game is in mixed strategies. The blind antisurface missile is programmed to perform a random maneuver sequence. The guidance law of the interceptor missile includes a bias, which partially compensates for the inability to achieve a satisfactory deterministic outcome and yields a nonzero probability of success. Moreover, the defense system must launch the interceptor missile at a randomly selected initial range from the incoming antisurface missile based on the solution of a game of timing. A new methodology is presented to assess the probability of successful interception as a function of the parameters of the scenario.

I. Introduction

SUCCESSFUL interception of antisurface missiles attacking high-value targets by defensive guided missiles, such as in ballistic missile defense and ship defense scenarios, presents an extreme challenge to the missile community. Tactical ballistic missiles, as well as modern antiship missiles, fly at very high speeds, and as a consequence their maneuvering potential is comparable to that of the interceptors. Successful interception of such a threat, carrying a lethal warhead, requires a small miss distance or even a direct hit. Although hit-to-kill performance against missiles flying at straight or ballistic trajectories was recently demonstrated,[1] new studies indicate that similar guidance accuracy cannot be achieved against highly maneuvering targets.[2,3]

In previous papers, the interception of antisurface missiles was formulated as an imperfect information, zero-sum, pursuit-evasion game with trajectory constraint imposed on the blind evader (the antisurface missile).[4–7] For the sake of analytical simplicity and generality, the analysis used linearized planar kinematics, simplified dynamic models, and nondimensional variables. The solution was based on earlier obtained results of the perfect information versions of the game. The unconstrained perfect information game[8] was motivated to analyze classical antiaircraft missile engagements (both air-to-air and surface-to-air) and served as the basis of several further studies,[9–12] such as extensions in the same context to three-dimensional space and imperfect information scenarios. A simplified analysis of a ship defense scenario in a game of timing formulation was presented in Ref. 5. Detailed solution of the perfect information game with constraint was presented in Ref. 13. It indicated that the constraint imposed on the evader becomes active and affects the solution only if the interception takes place at a rather short range from the target. Because in the majority of practically important scenarios (such as the interception of a tactical ballistic missile with a nonconventional warhead) this is not the case, the present paper concentrates on the unconstrained game model.

The objective is to extend the solutions presented in Refs. 5 and 7 by outlining a more general analysis of the imperfect information game. After formulating the missile vs missile interception scenario, the solution of the perfect information version of the game (the worst case from the defense point of view) is briefly reviewed. The analysis of the imperfect information game is nontrivial only if the perfect information game solution predicts that the guaranteed miss distance is larger than the lethal range of the interceptor warhead. A new methodology is then introduced that shows how to compute the probability of successful interception avoidance of the blind antisurface missile as a function of the scenario parameters. A detailed analysis of the corresponding game of timing is presented.

II. Problem Formulation

The analysis of the missile vs missile engagement is based on the following set of underlying assumptions.

1) The designated target T of the antisurface missile A, protected by the defense system, is stationary.

2) The engagement starts when the interceptor missile D is launched against A.

3) D has perfect information on A, but A has information only on T and no information on the position of D. The parameters of the engagement are known to all.

4) The interception of A by D must be completed within the maximum effective range of the defense system before A enters a prescribed safety zone, defined with respect to T.

5) If the interception fails, A hits and destroys T.

6) The conditions of the engagement are such that the trajectory constraints imposed on A (in order for it to reach the target T after avoiding interception) do not become active.

7) The engagement between the two missiles takes place in a plane.

8) Both missiles have constant velocities V_j and limited lateral accelerations $|a_j| < (a_j)_{\max}$, $j = A, D$.

9) A is assumed to have instantaneous dynamics, while the dynamics of D is expressed by a first-order transfer function with the time constant υ.

10) The trajectory of both missiles can be linearized about the initial line of sight.

Following assumptions 8 and 10, the time of the game (t_f) is determined by the ratio of the initial range (defined along the X axis) to the closing velocity. Based on assumptions 8 and 9, the equations of motion normal to the initial line of sight and the respective initial conditions are written as

$$\dot{y}_A = y_1, \qquad y_A(0) = 0 \tag{1a}$$

$$\dot{y}_1 = a_A^c, \qquad y_1(0) = y_{1_0} \tag{1b}$$

$$\dot{y}_D = y_2, \qquad y_D(0) = 0 \tag{1c}$$

$$\dot{y}_2 = y_3, \qquad y_2(0) = y_{2_0} \tag{1d}$$

Received July 22, 1996; presented as Paper 96-3881 at the AIAA Guidance, Navigation, and Control Conference, San Diego, CA, July 29–31, 1996; revision received Dec. 31, 1996; accepted for publication Feb. 28, 1997.

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$$\dot{y}_3 = \frac{a_D^c - y_3}{\upsilon}, \qquad y_3(0) = 0 \tag{1e}$$

where a_A^c and a_D^c are the commanded lateral accelerations of A and D, respectively,

$$a_A^c = (a_A)_{\max} v, \qquad |v| \le 1 \tag{2}$$

$$a_D^c = (a_D)_{\max} u, \qquad |u| \le 1 \tag{3}$$

The nonzero initial conditions y_{1_0} and y_{2_0} represent the respective initial velocity components not aligned with the line of sight. By assumption 10, these components are small compared to the components along the line of sight. Defining the pursuer/evader maneuver ratio

$$\mu \triangleq \frac{(a_D)_{\max}}{(a_A)_{\max}} \tag{4}$$

the normalized time-to-go

$$\theta(t) \triangleq (t_f - t)/\upsilon, \qquad \theta(0) \triangleq \theta_0 \tag{5}$$

and the normalized zero effort miss distance

$$Z(\theta) \triangleq \frac{(y_A - y_D) + \upsilon\theta(y_1 - y_2) - \upsilon^2 y_3(e^{-\theta} + \theta - 1)}{\upsilon^2 a_{A\max}} \tag{6}$$

and using these definitions in Eqs. (1) results in a one-dimensional differential game

$$\frac{\mathrm{d}Z}{\mathrm{d}\theta} = \mu(e^{-\theta} + \theta - 1)u - \theta v, \qquad Z(\theta_0) = Z_0 \tag{7}$$

The payoff function J of the engagement is the probability of successful interception-avoidance It is defined via a lethality function $\Phi(M)$, depending on the relationship between the normalized lethal radius of the warhead M^* and the normalized miss distance M and expressed as[14]

$$\Phi(M) = \begin{cases} 1 & \text{if} \quad M > M^* \\ 0 & \text{if} \quad M \le M^* \end{cases} \tag{8}$$

$$J = E[\Phi(M)] \tag{9}$$

Note that the definition of $\Phi(M)$, Eq. (8), is not an inherent feature of the ensuing analysis and can be replaced with any other lethality function.

The objective of D is to minimize this payoff function, i.e., to maximize the probability of successful interception, whereas the objective of A is to maximize it.

III. Perfect Information Game

In the perfect information game, the deterministic cost function is the normalized miss distance. This game[8] has only a single state variable Z with the dynamics described by Eq. (7). The solution is governed by a single parameter μ and is characterized by the existence of a minimal tube (see Fig. 1), which is reduced to a single point at $\theta = \theta_s(\mu)$, the nonvanishing solution of the equation

$$\theta = \mu(e^{-\theta} + \theta - 1) \tag{10}$$

For $\theta \le \theta_s$, as well as outside the minimal tube, the optimal strategies are bang-bang, i.e.,

$$u^*(\theta, Z) = v^*(\theta, Z) = \mathrm{sgn}(Z), \qquad Z \ne 0 \tag{11}$$

and the value of the game is a unique function of the initial conditions. The boundaries of the minimal tube are obtained by integrating Eq. (7) using Eq. (11). The segment $0 < \theta \le \theta_s$ of the θ axis ($Z = 0$) is a dispersal line of the evader A, where the optimal maneuver can be either to the left or to the right, by a random selection. Inside the minimal tube, in the region where $\theta > \theta_s$ (denoted as $\mathcal{D}_0$), the optimal strategies are arbitrary. All of the trajectories starting in $\mathcal{D}_0$

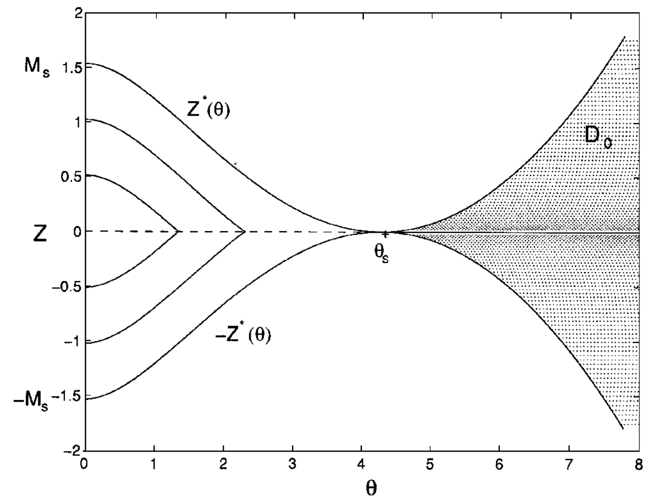

Fig. 1 Solution of the unconstrained perfect information game.

reach the point ($\theta = \theta_s$, $Z = 0$), and the value of the game (the guaranteed normalized miss distance from $\mathcal{D}_0$) is constant:

$$J_0^* = \theta_s - (\mu - 1)\theta_s^2/2 = M_s(\mu) \tag{12}$$

The major conclusion drawn from this analysis has been that if μ is sufficiently large ($\mu > 2$, at least), then the guaranteed miss distance is negligibly small. In future missile vs missile engagements, such a favorable maneuver ratio may not exist. Therefore, the results of Ref. 8 should be reexamined for smaller values of μ. For $\mu < 1$, $\mathcal{D}_0$ does not exist. In this case, the values of M_s have to be obtained by integrating Eq. (7) directly with the appropriate θ_0. In Ref. 5, it was shown that very large miss distances, unacceptable for antimissile defense, are obtained for $\mu \to 1$.

Depending on the value of M^* (the nondimensional lethal radius of the warhead of D) the saddle-point value of the miss distance may or may not be satisfactory for an effective defense. Because in reality A has no information on D, in most cases the actual miss distance will be smaller, as analyzed in the next section.

IV. Imperfect Information Game Analysis

Heuristic Strategies

In this section, it is assumed from the outset that the parameters of the scenario are such that the ratio $\eta \triangleq M^*/M_s$ is unsatisfactory for the defense ($M^* < M_s \Rightarrow \eta < 1$). Otherwise the entire analysis is irrelevant.

The information structure of the game is asymmetrical. It is assumed that the defense system has perfect information on the positions of both A and D but A knows only its own position with respect to T. The fixed parameters of the engagement are assumed to be known to all.

Without information on the position of D, an optimal evasion of A, in the deterministic sense, cannot be performed. Thus, A must maneuver randomly using a mixed strategy, inasmuch as a straight-line flight or a maneuver to a fixed direction is predictable. The results of the perfect information game[8] indicate that for achieving the guaranteed miss distance M_s there must be a maximal maneuver to the same direction for a duration of at least θ_s before the interception.

These observations lead to the following intuitive guidelines for the structure of the evasive maneuvers.

1) The maneuvering sequence should cover the entire interception range of the defense system using maximum lateral acceleration.

2) The optimal sequence must employ a small number of randomly timed direction changes (switches).

3) The duration of each maneuver should be of the order of θ_s.

In summary, the parameters of a pure evasive strategy are the switching distances from T. The random selection of the parameters creates the mixed strategy of A.

In this situation, the defense system, in spite of having perfect information, must select the time for launching D randomly, covering the entire feasible domain (otherwise A, knowing the interception range, can plan a deterministic optimal evasion). The best launch direction, in a perfect information scenario, is toward the predicted collision point. In the imperfect information scenario of interest ($\eta < 1$ and random maneuvers of A) a nonzero initial bias, based on a presumed continuous maneuver, may be considered. The magnitude of this bias and its direction can be random. Thus, the defense also applies a mixed strategy.

Solution Methodology

There exists no general solution methodology for imperfect information games in a continuous state space. Discretization of the state space does lead to a numerical algorithm, the complexity of which becomes prohibitive as the number of partition cells increases. A novel approach, developed first in Ref. 11 and extended to games of timing in Ref. 14, is presented herein for the analysis of missile vs missile interception scenarios with imperfect information.

Pure Strategies

The first stage of the solution is to define a pure strategy set for each player. One can discretize the parameter space of these pure strategy sets and create a matrix game in which different combinations of strategies of D are mapped into different rows and different combinations of strategies of A are mapped into different columns. Each element of this matrix corresponds to a unique combination of a pure strategy pair in the discretized game space. The outcome of a game scenario, played with such a combination of pure strategies, is the probability of a successful interception avoidance as expressed by Eq. (9). By solving this matrix game with sufficiently fine discretization, one can closely approximate the optimal distribution over the parameter space of both players, i.e., their optimal mixed strategy.

Based on the guidelines mentioned earlier, the pure strategy structure of the two players will consist of the following.

A will fly toward its designated target and at a selected time will start its maneuver sequence. This sequence will consist of maximal acceleration maneuvers with randomly timed direction changes. It is assumed that the random duration δ between any two consecutive direction changes (switches) of the maneuver has a well-defined distribution function F_δ [and a corresponding probability density function (PDF) p_δ, if it exists].

D will select the moment for launching the interceptor, the direction of the launch, and a guidance law out of n optional biased guidance laws. The minimal number of biased guidance laws needed to cover the entire reachable set (possible final locations) of A can be computed. It is assumed that each biased guidance law is a modification of Eq. (11), derived from the perfect information game solution by inserting a time-varying bias function $\{b_i(\theta)\}_{i=1}^n$:

$$u_i = \mathrm{sgn}[Z - b_i(\theta)] \tag{13}$$

The launch direction, compatible with each guidance law, eliminates the argument of the sign function in Eq. (13) at θ_0,

$$Z(\theta_0) = b_i(\theta_0) \tag{14}$$

For any pair of interceptor guidance law and maneuver sequence of the antisurface missile, there exists a game of timing,[5] as described in the sequel. In this work, it is proposed to separate the analysis of the end game from that of the game of timing, thus rendering the computation of the probability of a successful interception avoidance manageable. Solving the end game and the game of timing together would have created a very large matrix game, requiring a brute force solution. In addition to an excessive computational burden, the insight gained by separating the two different games would have been lost.

Stochastic Process

Define θ_m as the normalized time-to-go of the evader's first maneuver. For any arbitrary values of θ_0 and θ_m, F_δ and $b_i(\theta)$ determine a stochastic process whose sample functions are trajectories in the game space (θ, Z). These trajectories are composed of segments, denoted as $Z[\theta; (\theta_i, Z_i)]$, where (θ_i, Z_i) is the segment starting point in the game space and θ is a running parameter along this segment.

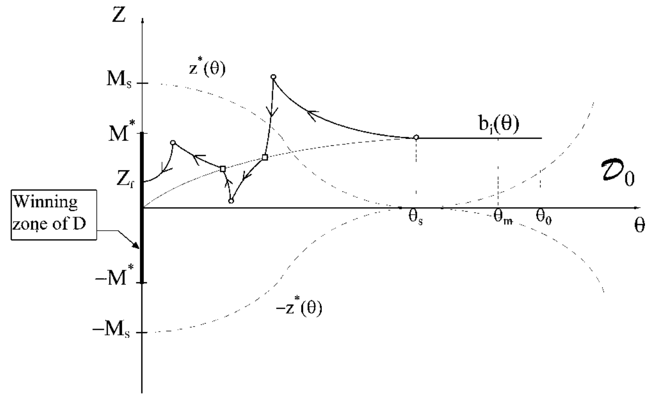

Fig. 2 Stochastic process.

A sample trajectory is schematically depicted in Fig. 2, where it is assumed that the slope of $b_i(\theta)$ is zero for $\theta \geq \theta_s$. The trajectory starts on the switching line of D at θ_0 and terminates at $\theta = 0$. Assuming that $\mu > 1$, then for $\theta_0 > \theta_s$ (which corresponds to the trajectory depicted in Fig. 2), D can keep the trajectory on the switching line, independently of the maneuvers of A, as long as $\theta > \theta_s$. If the maneuvering sequence starts at $\theta_m > \theta_s$, the trajectory leaves the switching line of D at θ_s toward the direction of the last maneuver. Otherwise, the trajectory may leave the switching line of D at θ_m. This first maneuver (always away from the switching line of D) is along a trajectory segment, defined as $\tilde{Z}_1(\theta) \triangleq \tilde{Z}[\theta; (\theta_d, b_i(\theta_d))]$, where θ_d is the normalized time-to-go at which the trajectory departs from the switching line. Notice that this trajectory is generated by using the strategy pair $u_i = v_i = \mathrm{sgn}[Z - b_i(\theta)]$ in Eq. (7); hence, it is parallel to an optimal trajectory of the perfect information game.

Because A maneuvers randomly, an additional switch in its control can occur at some $\theta = \eta > 0$. The new trajectory segment generated by the strategy pair $u_i = -v_i = \mathrm{sgn}[Z - b_i(\theta)]$ is denoted by $\hat{Z}_1(\theta) \triangleq \hat{Z}[\theta; (\eta, \tilde{Z}_1(\eta))]$. If A switches again at $\theta = \zeta$ (before the trajectory crosses the switching line of D), a new segment, defined as $\tilde{Z}_2(\theta) \triangleq \tilde{Z}[\theta; (\zeta, \hat{Z}_1(\zeta))]$, is created. Whenever a trajectory of the type $\hat{Z}$ crosses the switching line of D, the switch of control by D creates a trajectory segment of the type $\tilde{Z}$.

To compute the lethality function [Eq. (8)] corresponding to a specific sample of the stochastic process, the terminal surface of the game ($\theta = 0$) is divided into two parts: the winning zone of D, characterized by $M \leq M^*$ (see Fig. 2), and the winning zone of A, where $M > M^*$.

Any sample trajectory, passing through some switching point of A, (θ_k, Z_k), ends at a randomly distributed point $[\theta = 0, Z = Z_f(\theta_k, Z_k)]$ (which constitues an implied definition of Z_f). If at that switching point a trajectory segment of the type $\tilde{Z}$ is generated, the end point of the sample trajectory is denoted by $\tilde{Z}_f(\theta_k, Z_k)$. In an analogous manner, $\hat{Z}_f(\theta_k, Z_k)$ is defined as the endpoint of the sample trajectory if at the switching point a trajectory segment of the type $\hat{Z}$ is generated.

For any switching point (θ, Z) of A in the game space, the following four events are defined. Let $\mathcal{A}$ be the event that A will not perform any additional switch until the end of the game, i.e.,

$$\mathcal{A} \triangleq \{\delta(\theta) \geq \theta\} \tag{15}$$

where $\delta(\theta)$ is the duration of the maneuver starting at the switching point (θ, Z). $\mathcal{B}$ is defined as the event that A will perform at least one additional switch until the end of the game, i.e.,

$$\mathcal{B} \triangleq (\mathcal{A})^c \tag{16}$$

Let $\mathcal{C}$ be the event that at the point (θ, Z) a trajectory segment of the type $\tilde{Z}$, which ends within the winning zone of A, is generated, i.e.,

$$\mathcal{C} \triangleq \{\Phi[|\tilde{Z}_f(\theta, Z)|] = 1\} \tag{17}$$

Analogously, $\mathcal{D}$ is defined as

$$\mathcal{D} \triangleq \{\Phi[|\hat{Z}_f(\theta, Z)|] = 1\} \tag{18}$$

For each such switching point (θ, Z), two probability measures P_0 and Q_0 are defined:

$$P_0(\theta, Z) \triangleq \text{Prob}\{\mathcal{C}\} \tag{19}$$

$$Q_0(\theta, Z) \triangleq \text{Prob}\{\mathcal{D}\} \tag{20}$$

In the sequel, two coupled integral equations whose solution comprises the probability measures $P_0(\theta, Z)$ and $Q_0(\theta, Z)$ are derived.

Consider first the event $\mathcal{C}$. Using the total probability theorem yields

$$\text{Prob}\{\mathcal{C}\} = \text{Prob}\{\mathcal{C} \mid \mathcal{A}\}\text{Prob}\{\mathcal{A}\} + \text{Prob}\{\mathcal{C} \mid \mathcal{B}\}\text{Prob}\{\mathcal{B}\} \tag{21}$$

because $\mathcal{A}$ and $\mathcal{B}$ comprise a partition. Outruling concentrated mass probability at $\delta = \theta$ for practical reasons yields

$$\text{Prob}\{\mathcal{A}\} = 1 - F_\delta(\theta) \tag{22}$$

Suppose that no additional switch has occurred until the end of the game. In this case, the (deterministic) endpoint of the trajectory, denoted by $\tilde{Z}_f^{\mathcal{A}}(\theta, Z)$, can be computed. Therefore,

$$\text{Prob}\{\mathcal{C} \mid \mathcal{A}\} = \Phi\Big[\big|\tilde{Z}_f^{\mathcal{A}}(\theta, Z)\big|\Big] \tag{23}$$

To compute the second term in Eq. (21), the time interval $[0, \theta]$ is divided into N equal subintervals of the size

$$\Delta\,\theta = \theta / N \tag{24}$$

The event $\mathcal{B}$ can be partitioned into N disjoint subevents

$$\mathcal{B} = \bigcup_{i=1}^{N} \mathcal{B}_i \tag{25}$$

Here, $\mathcal{B}_i$ is the event that the first additional switch has occurred at the point $[\bar{\eta}_i, \tilde{Z}(\bar{\eta}_i)]$ {here $\tilde{Z}(\bar{\eta}_i)$ is a short notation for $\tilde{Z}[\bar{\eta}_i; (\theta, Z)]$}, where $\bar{\eta}_i$ belongs to the ith subinterval, i.e.,

$$\theta - i\Delta\,\theta < \bar{\eta}_i \le \theta - (i-1)\Delta\,\theta, \qquad i = 1, \ldots, N \tag{26}$$

Because $\delta = \theta - \bar{\eta}_i$, for large N, $\text{Prob}\{\mathcal{B}_i\}$ can be approximated by

$$\text{Prob}\{\mathcal{B}_i\} \approx p_\delta(\theta - \bar{\eta}_i)\Delta\,\theta \tag{27}$$

Additionally, using the definition of Q_0 [Eq. (20)] yields

$$\text{Prob}\{\mathcal{C} \mid \mathcal{B}_i\} = Q_0[\bar{\eta}_i, \tilde{Z}(\bar{\eta}_i)] \tag{28}$$

Accordingly,

$$\text{Prob}\{\mathcal{C} \mid \mathcal{B}\}\text{Prob}\{\mathcal{B}\} \approx \sum_{i=1}^{N} Q_0[\bar{\eta}_i, \tilde{Z}(\bar{\eta}_i)]p_\delta(\theta - \bar{\eta}_i)\Delta\,\theta \tag{29}$$

Taking the limit $N \to \infty$ and $\Delta\,\theta \to 0$ and substituting Eqs. (22), (23), and (29) in Eq. (21) yields the following integral equation:

$$P_0(\theta, Z) = [1 - F_\delta(\theta)]\Phi\Big[\big|\tilde{Z}_f^{\mathcal{A}}(\theta, Z)\big|\Big] + \int_0^\theta p_\delta(\theta - \eta)Q_0[\eta, \tilde{Z}(\eta)]\,\mathrm{d}\eta \tag{30}$$

The same reasoning can be used to derive the analogous integral equation for the probability measure Q_0,

$$Q_0(\theta, Z) = [1 - F_\delta(\theta)]\Phi\Big[\big|\hat{Z}_f^{\mathcal{A}}(\theta, Z)\big|\Big] + \int_0^\theta p_\delta(\theta - \zeta)P_0[\zeta, \hat{Z}(\zeta)]\,\mathrm{d}\zeta \tag{31}$$

where $\hat{Z}(\zeta)$ is defined analogously to $\tilde{Z}(\eta)$.

Notice that whenever the trajectory segments cross the switching line of D, the Q_0 and P_0 under the integrals of Eqs. (30) and (31) will be replaced by P_0 and Q_0, respectively.

For any given probability distribution of the random maneuver sequence, F_δ, lethality function $\Phi(|Z_f|)$ and location of the (biased) switching line of D, $b_i(\theta)$, the values of P_0 and Q_0 at any point in the game space can be found via a solution of the coupled integral equations (30) and (31).

In the analysis presented so far, the functions $P_0(\theta, Z)$ and $Q_0(\theta, Z)$ were computed referring to switching points of A. At a point (θ, Z), which is not a switching point of A, the probability of A to reach its winning zone depends also on the normalized time elapsed from the last switch of A, denoted by Δ. In this case, the probability of success of A is denoted by $P_\Delta(\theta, Z)$ if the point (θ, Z) is on a trajectory of the type $\tilde{Z}$, or by $Q_\Delta(\theta, Z)$ if the trajectory is of the type $\hat{Z}$. The computation of $P_\Delta(\theta, Z)$ and $Q_\Delta(\theta, Z)$ is discussed in Appendix A.

If $\theta_0 > \theta_s$, D can guarantee that all trajectories will go through the same point, determined by the biased guidance law, at θ_s. Therefore, in this case, which characterizes the majority of scenarios of interest, the probabilities P_Δ and Q_Δ have to be computed only at that point.

Mapping of the Matrix Game

Any cell of the matrix game represents a combination of a particular probability distribution of the random maneuvering sequence, a specific biased guidance law, and a pair (θ_0, θ_m). Each such cell has to be associated with an adequate value of the payoff function. Based on the actual information pattern two basically different situations can be distinguished. If D knows the direction of the current maneuver of A, it can launch toward an appropriate biased direction, thus enhancing its probability of success. In this case, the payoff of the matrix cell is Q. If, however (as can be generally expected), D knows only the current position of A but has no information on the direction of its maneuver, the selection of the biased switching line has to be random and the resulting probability of interception avoidance is the weighted average of the respective values of P and Q. Examples to both cases will be given in the sequel.

If A starts its maneuver sequence after the launch of D, i.e., $\theta_m \le \theta_0$, the values of P_0 and Q_0 can be computed using $[\theta_m, b_i(\theta_m)]$ as the starting point of the stochastic process, as detailed in the preceding subsection. If, however, $\theta_m > \theta_0$, then the avoidance probabilities of A, denoted in this case as $\bar{P}_\Delta[\theta_0, b_i(\theta_0)]$ and $\bar{Q}_\Delta[\theta_0, b_i(\theta_0)]$ should be computed by averaging over the random time Δ, elapsed from the last switch of A until the launch, as shown in Appendix B.

V. Example: Random Telegraph Maneuver

In this section, it is assumed that the random maneuver sequence of A is of the random telegraph type (frequently used in missile analysis[15]). The probability distribution of this maneuver sequence is $F_\delta(\alpha) = 1 - \exp(-\lambda\alpha)$. The corresponding stochastic process is characterized by the single parameter λ. The average duration between any two subsequent direction changes (switches) is $1/\lambda$, and the probability that there will be no switch during a given period of time $\Delta\,t$ is equal to $\exp(-\lambda\Delta\,t)$. This probability is independent of any past event, i.e., the process has no memory. Thus, for such a maneuver, the probabilities P and Q do not depend on the time elapsed from the last switch of A and, therefore, $P \equiv P_0 \equiv P_\Delta$ and $Q \equiv Q_0 \equiv Q_\Delta$. Three different interceptor guidance strategies are considered in the sequel.

Unbiased Interceptor Guidance

Assume first that 1) the guidance law of the interceptor is the one of the perfect information game,[8] 2) the evader acceleration is not estimated, and 3) the initial conditions are inside the minimal tube. In this case, the switching line of D is $Z = 0$. For this case, lines of constant probability of interception avoidance are shown in Fig. 3 for $\lambda\theta_s \triangleq \lambda_s = 2.0$, $\eta = 0.54$, and $\mu = 1.3$. Because both functions are symmetrical with respect to the switching line $Z = 0$, lines of constant P are plotted in the upper-half of the game space $(Z > 0)$ and of constant Q in the lower-half $(Z < 0)$. The dotted lines represent the optimal trajectories resulting in M_s (see Fig. 1). Because the majority of practical initial conditions are inside the

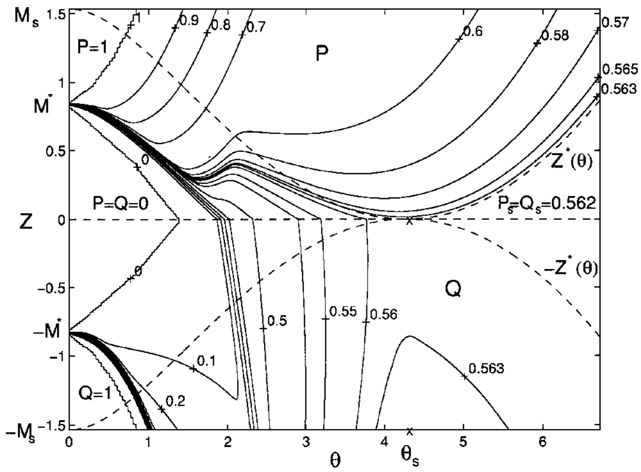

Fig. 3 Lines of constant probability of interception avoidance without bias, λ_s = 2.0 and μ = 1.3.

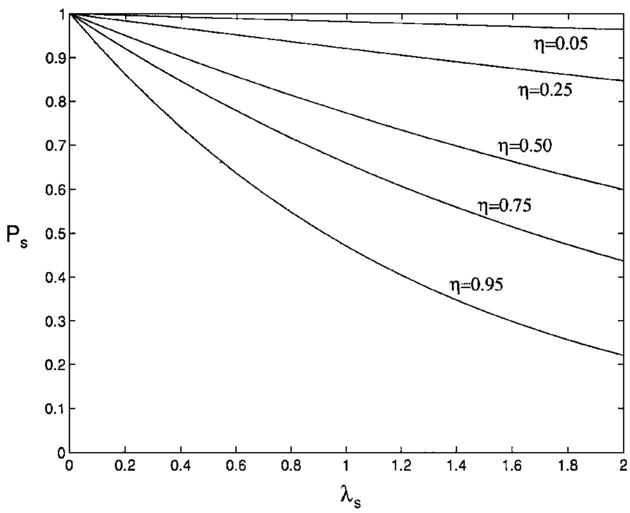

Fig. 4 Value of the imperfect information game without bias.

minimal tube with $\theta > \theta_s$, i.e., in $\mathcal{D}_0$, the most important result is the value of $P_s \triangleq P(\theta_s, Z = 0)$, representing the guaranteed probability of interception avoidance (in the example, $P_s = 0.562$). This is the value of the imperfect information game, assuming that the interceptor uses the deterministic guidance law of Ref. 8. This value depends on the engagement parameters μ, η, and λ_s, as plotted in Fig. 4 for the fixed value of $\mu = 1.3$. Two major conclusions can be drawn from these results.

1) The interest of A is to use a value of λ_s as small as possible. Practical considerations prohibit $\lambda = 0$, and a lower bound of $\lambda_{s\,\min}$ has to be established, such that the probability of hitting the real trajectory constraint (which is neglected by assumption 6) will be negligible.

2) In spite of the information advantage and even for high values of $\eta < 1$, the probability of a defense failure using the deterministic guidance law of Ref. 8 reaches very large values.

Biased Interceptor Guidance

The probability of a defense failure can be reduced by using a bias either in the initial launch direction and/or in the guidance law. If $0.5 < \eta < 1$, then it is easy to find a bias that guarantees that every second antisurface missile, on average, will be intercepted and destroyed or that two interceptors fired against the same target with different biases will guarantee the protection of a defended site. Without being able to estimate the accelerations of A, the bias must be selected randomly and the guidance law must also include a compatible time varying bias $b_i(\theta)$.

The objective of the bias in the guidance law is to achieve a certain successful interception if A maneuvers in the correctly assumed direction and to maximize its probability even if a direction change occurs. In Ref. 14 the following time-varying biases were selected:

$$b_1(\theta) = \tfrac{1}{2}\theta^2(1-\mu) + \mu(e^{-\theta} + \theta - 1) - 0.8M^* \qquad (32a)$$

$$b_2(\theta) = -b_1(\theta) \qquad (32b)$$

where the coefficient of 0.8 provides 20% of safety margin. The two biased switching lines are parallel to the boundaries of the minimal tube. They intersect twice on the θ axis, at $\theta = \theta^*$ and $\theta = \theta_B$, as can be seen in Fig. 5. The last time to select the sign of the bias is $\theta_B(\eta, \mu) > \theta_s$. The existence of the bias modifies the avoidance probability functions $P(\theta, Z)$ and $Q(\theta, Z)$. The values of $P_B \triangleq P(\theta_B, Z = 0)$ and $Q_B \triangleq Q(\theta_B, Z = 0)$ are not equal to each other ($Q_B < P_B$). If the selection between b_1 and b_2 is performed randomly with equal probabilities, the relevant payoff function is

$$\hat{P} \triangleq \frac{P_B + Q_B}{2} \qquad (33)$$

Using Estimated Evader Acceleration

If the defense system can accurately estimate the lateral accelerations of A, a better result can be expected. Using the information on the acceleration of A obtained before θ_B, the interceptor can determine the direction of the correct bias by $\mathrm{sgn}(v)$ and reduce the probability of interception avoidance to Q_B. Assuming an ideal estimation, the value of Q_B is plotted in Fig. 6. It can be seen that, for a given η, there exists a value of $\lambda_s = \lambda_s^*$ that maximizes Q_B.

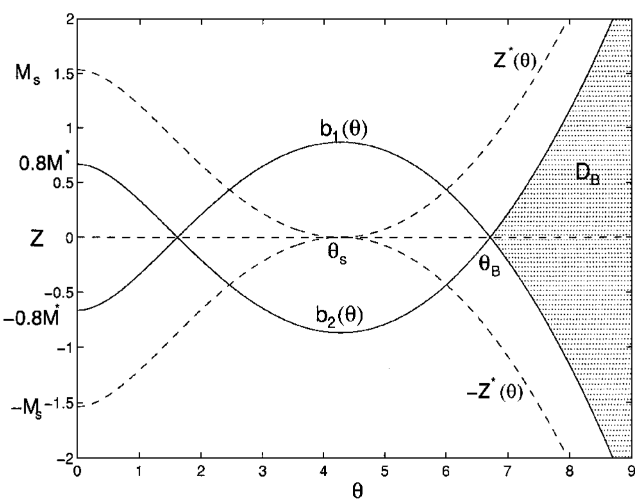

Fig. 5 Biased switching lines, $b_1(\theta)$ and $b_2(\theta)$, in the imperfect information game.

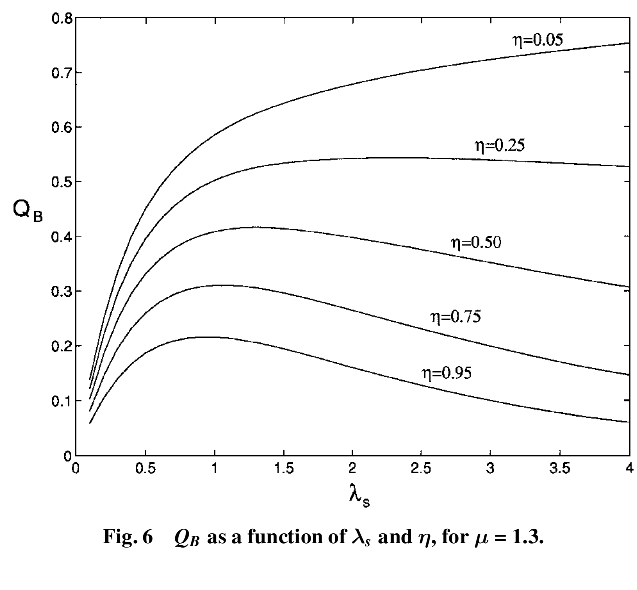

Fig. 6 Q_B as a function of λ_s and η, for μ = 1.3.

A comparison between the different cases with these parameters and with $\lambda_{s\,\min} = 0.3$ has shown that the success probability of A is 0.92 against a guidance without bias and 0.48 against a guidance with a random bias (in both cases, $\lambda_s^* = \lambda_{s\,\min} = 0.3$). When a guidance with the correct bias is used, the probability of success for A is reduced to 0.39, and the optimal value of λ_s of the maneuver is then in the range $1.0 < \lambda_s^* < 1.5$.

It is of common experience that acceleration estimation is typically characterized by a slow convergence rate. Any delay in identifying the correct direction of the target acceleration increases the probability of interception avoidance, though it will always remain lower than in the case of a random guess. The information delay also reduces the optimal value of λ_s^*.

VI. Game of Timing

In the preceding section an end-game analysis was carried out, facilitating the evaluation of the probability of successful interception avoidance by random maneuvering. These results show that the probability of interception avoidance apparently assumes three different (constant) values, depending on the length of the end game, thus justifying the proposed separation of the analysis into two phases. The point of reference for the analysis presented in that section is the time of closest approach ($\theta = 0$), determined by the randomly selected time of launching the interceptor missile. Moreover, the randomly selected starting time of the maneuver sequence is assumed to be given. As mentioned earlier, the optimal selection of the time for these actions is the solution of a game of timing, presented in this section.

In this section, it is assumed that A uses exactly the same random telegraph maneuver sequence with $\lambda = \lambda^*$ (as in Sec. V) and D applies a biased guidance law, based on an instantaneous identification of the maneuver direction of A. For illustrative purposes and for the sake of simplicity, a point defense scenario is considered.

The randomly selected time of launching D determines the duration θ_0 of the game. The probabilistic outcome of the end game is jointly determined by θ_0 and the random starting moment θ_m of the maneuver sequence of A.

Depending on the randomly selected values of θ_0 and θ_m, three different outcomes are possible.

1) The maneuver sequence of A has started too late and, therefore, the resulting miss distance is too small ($M \leq M^*$). Thus, the probability of a successful interception avoidance is zero.

2) The maneuver sequence has started near the time of closest approach. The maneuver can generate sufficiently large miss distances $M > M^*$, but the time left for using the estimated maneuver for the selection of the appropriate guidance law is not sufficient ($\theta_B > \theta_m$). In this case, the probability of successful interception avoidance is $\hat{P} = (P_B + Q_B)/2$.

3) If the time suffices for the selection of the appropriate biased guidance law ($\theta_B \leq \theta_m$), the probability of successful interception avoidance is Q_B.

This situation can be formulated as a classical game of timing, as shown in Fig. 7. The designer of the antisurface missile preprograms the range X_{Am} from the surface target T where the maneuver sequence should start. The defense system selects to launch the interceptor when the antisurface missile reaches the range X_{A0} and determines in this way the duration of the engagement, as well as $X_f = \theta_0 \upsilon V_D$, the distance where the interception will take place. Obviously, the time for launching D has to be selected randomly to satisfy

$$R_{\min} \leq X_f \leq R_{\max} \tag{34}$$

where $R_{\min}$ and $R_{\max}$ are determined by the performance of the interceptor missile and safety considerations. If these limits are known also to the opponent, the extremal values for selecting X_{Am} will be

$$X_{A\min} = R_{\min} \tag{35}$$

and

$$X_{A\max} = R_{\max} + \upsilon\theta_s V_A \tag{36}$$

(Note that larger ranges than $X_{A\max}$ will not yield any benefit because interception cannot take place beyond $R_{\max}$. Starting to maneuver earlier would only increase the detectability of maneuver direction.) The three zones of different outcome are displayed in Fig. 7.

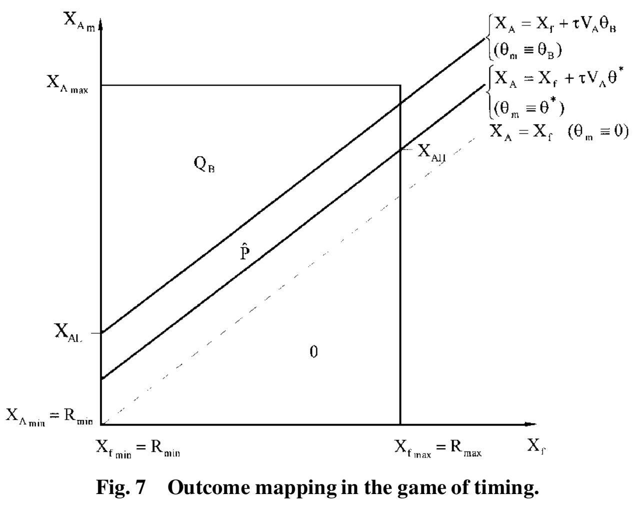

Fig. 7 Outcome mapping in the game of timing.

The classical methodology for solving game of timing is outlined in Ref. 16. We will use the concept of equalizer strategies described there. A mixed strategy is called an equalizer strategy if it guarantees the same outcome against each possible strategy of the opponent. If one can find equalizer strategies for both players with the same outcome, the game of timing is solved. The constant outcome is the value of the game.

Because $Q_B < \hat{P}$, the objective of A is to maximize the probability of being in the small zone where the probability of successful interception avoidance is $\hat{P}$. Therefore, it will select X_{Am} only in the region $X_{AL} < X_{Am} < X_{AH}$ where such an outcome has a high probability. It has to divide this region to the smallest number n of subregions such that, for any selection of X_f by D, at least one out of the n possible selections of A will result in an outcome of $\hat{P}$. In Fig. 8, an example of four subregions is shown.

In general, A will select one out of n possible elements $\{X_{Ami}\}_{i=1}^{n}$. Let A select the subregion i with probability α_i. If D selects X_{f1} (see Fig. 8), the outcome for A will be

$$J_A = \alpha_1 \hat{P} \tag{37}$$

If D selects X_{f2}, the outcome will be

$$J_A = \alpha_1 Q_B + \alpha_2 \hat{P} \tag{38}$$

and, in general, if D selects X_{fi}, the outcome will be

$$J_A = (\alpha_1 + \alpha_2 + \cdots + \alpha_{i-1}) Q_B + \alpha_i \hat{P} \tag{39}$$

Equating all of the outcomes leads to the following recursive relationship:

$$\alpha_{i+1} = q\alpha_i \tag{40}$$

where

$$q = 1 - (Q_B / \hat{P}) \tag{41}$$

With the normalization on

$$\alpha_i \left(\sum_{i=1}^{n} \alpha_i = 1 \right)$$

this leads to

$$\alpha_1 = \frac{1 - q}{1 - q^{n+1}} \tag{42}$$

and

$$J_A = \frac{Q_B}{1 - q^{n+1}} \tag{43}$$

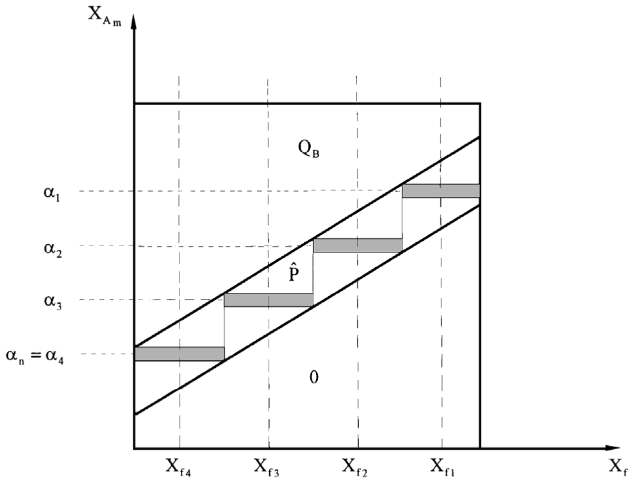

Fig. 8 Selection of segments by the attacker.

This mixed strategy for A will be optimal if there is also an equalizer strategy for D yielding the same guaranteed outcome. Assume that D also divides the domain of the admissible launching ranges [Eq. (34)] into a maximum number of subregions, such that, for any selection of X_{Am} by A, the outcome will be $\hat{P}$ only in a single subregion at most. Such a decomposition is shown in Fig. 9. The probability of D to select any subregion j is denoted by δ_j (the probability of selecting the shortest interception range being δ_1). In a similar manner to Eqs. (37–39), the explicit equations for an equalizing strategy of D are

$$J_D = \delta_1 \hat{P} \tag{44}$$

$$J_D = \delta_1 Q_B + \delta_2 \hat{P} \tag{45}$$

$$J_D = (\delta_1 + \delta_2 + \cdots + \delta_{j-1}) Q_B + \delta_j \hat{P} \tag{46}$$

yielding, in an analogous manner,

$$\delta_{j+1} = q\delta_j \tag{47}$$

$$\delta_1 = \frac{1-q}{1-q^{n+1}} = \alpha_1 \tag{48}$$

and the guaranteed outcome for D, which becomes the value of the game, is

$$J_D = \frac{Q_B}{1-q^{n+1}} = J_A = J^* \tag{49}$$

The number of subregions n to be selected by both sides is the largest integer, which is still smaller than

$$n' = \frac{R_{\max} - R_{\min}}{\Delta X_A} \tag{50}$$

where

$$\Delta X_A \triangleq \upsilon V_A(\theta_B - \theta^*) \tag{51}$$

If n' is not an integer, the subregions overlap. The width of each subregion and the selection inside the subregions is arbitrary.

If the maneuvering sequence starts at or before $X_{A\max}$ [given by Eq. (36)], the worst result that A can achieve is bounded by Q_B. As can be observed from Eq. (49), the difference between this outcome and the optimal value of the game, J^*, is rather small depending on the numerical values of q and n.

If, instead of the optimal mixed launch strategy [given by $\delta^* = (\delta_1, \ldots, \delta_n)$ in Eqs. (47) and (48)], the defense selects a uniform probability distribution, the outcome is bounded by

$$J_U \leq Q_B + \frac{\hat{P} - Q_B}{n} \tag{52}$$

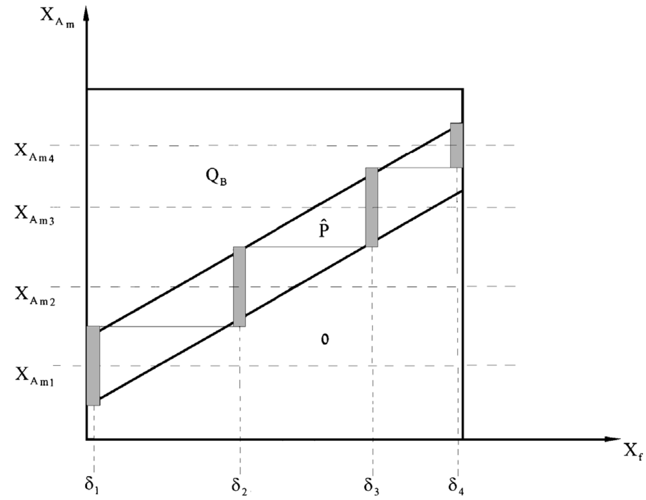

Fig. 9 Selection of segments by the defender.

For an illustrative numerical example, the same data as in the preceding section are used. These data give $n = 4$, $Q_B(\lambda_s = 1.5) = 0.393$, and $\hat{P} = 0.408$. Substitution in Eq. (49) results in $J_A = (1 + 6.7 \times 10^{-8}) Q_B$. Obviously, this result is very close to Q_B.

Substituting the data in Eq. (52) yields $J_U \leq 0.396 = 1.0096 Q_B$. Again, this result approximates Q_B very closely, and thus with the data of the given example, both players can use simple strategies for the selection of their ranges (early maneuvering of A and uniformly distributed launch time), without obtaining practically worse results.

VII. Conclusions

A new methodology is presented for analyzing future missile vs missile scenarios, for which a deterministic investigation predicts an unsatisfactory defense performance. In this case, both sides (the designer of the antisurface missile as well as the defense system) must use random (mixed) strategies. A generalized approach is outlined for computing the probability of successful interception (or interception avoidance) as a function of the scenario parameters. This approach may be applied also to other differential games with similar information pattern. Using an example, it is demonstrated that the computation becomes easily manageable if the random maneuvering sequence of the antisurface missile is assumed to be of a random telegraph type.

The new methodology creates a fresh insight into the complex nature of this very critical problem. It also generates a powerful research potential to evaluate design and operational tradeoffs for currently designed, as well as future antimissile defense systems. Future analyses facilitated by the new methodology include: detailed parametric analysis of future antimissile defense scenarios, sensitivity assessment of the homing performance of currently developed defense systems to antisurface missile maneuvers, and development of efficient defense strategies against unpredictably maneuvering antisurface missiles.

Appendix A: Derivation of P_Δ and Q_Δ

When (θ, Z) is not a switching point of A, the normalized time Δ, which elapsed from the last switch of A, determines the avoidance probability measures $P_\Delta(\theta, Z)$ and $Q_\Delta(\theta, Z)$.

Let γ denote the duration from the current normalized time-to-go θ until the next switch, i.e., $\gamma \triangleq \delta(\theta + \Delta) - \Delta$, where $\delta(\theta + \Delta)$ is defined analogously to the definition implied in Eq. (15). Notice that the definition of Δ yields $\text{Prob}\{\gamma \leq 0\} = 0$; hence, the probability distribution of γ is

$$F_\gamma(\alpha) = \text{Prob}\{\gamma \leq \alpha \mid \gamma > 0\} = \text{Prob}\{\delta \leq \alpha + \Delta \mid \delta > \Delta\}$$
$$= [1 - F_\delta(\Delta)]^{-1}[F_\delta(\alpha + \Delta) - F_\delta(\Delta)] \tag{A1}$$

and, by differentiation, its PDF is

$$p_\gamma(\alpha) = [1 - F_\delta(\Delta)]^{-1} p_\delta(\alpha + \Delta) \tag{A2}$$

Replacing in Eqs. (30) and (31) the functions p_δ and F_δ by p_γ and F_γ, respectively, yields

$$P_\Delta(\theta, Z) = [1 - F_\delta(\Delta)]^{-1}\left\{[1 - F_\delta(\theta+\Delta)]\Phi\left[\left|\tilde{Z}_f^A(\theta, Z)\right|\right] + \int_0^\theta p_\delta(\theta+\Delta-\eta)Q_0[\eta, \tilde{Z}(\eta)]\,d\eta\right\} \tag{A3}$$

$$Q_\Delta(\theta, Z) = [1 - F_\delta(\Delta)]^{-1}\left\{[1 - F_\delta(\theta+\Delta)]\Phi\left[\left|\hat{Z}_f^A(\theta, Z)\right|\right] + \int_0^\theta p_\delta(\theta+\Delta-\zeta)P_0[\zeta, \hat{Z}(\zeta)]\,d\zeta\right\} \tag{A4}$$

In these equations, the terms $\tilde{Z}_f^A(\theta, Z)$, $\hat{Z}_f^A(\theta, Z)$, $\tilde{Z}(\eta)$, and $\hat{Z}(\zeta)$ have the same meaning as in Sec. IV except that here the point (θ, Z) (referred to in their definitions) is not a switching point.

Appendix B: Derivation of $\bar{P}_\Delta$ and $\bar{Q}_\Delta$

Let σ denote the time interval from the beginning of the maneuver sequence to the launch moment, i.e.,

$$\sigma \stackrel{\Delta}{=} \theta_m - \theta_0 \tag{B1}$$

The time elapsed from the last switch of A prior to the launch moment Δ is a mixed random variable. To determine its probability distribution function, the following possibilities are considered.

1) No additional switch has occurred until the launch moment, i.e., $\Delta = \sigma$. The probability of this event is

$$\text{Prob}\{\Delta = \sigma\} = \text{Prob}\{\delta \geq \sigma\} = 1 - F_\delta(\sigma) \tag{B2}$$

(assuming, as before, that $\text{Prob}\{\delta = \sigma\} = 0$). In this case,

$$P_\Delta \equiv P_\sigma, \qquad Q_\Delta \equiv Q_\sigma \tag{B3}$$

2) A switched at least one additional time prior to the launch moment, i.e., $0 \leq \Delta < \sigma$. In this case, p_Δ, the PDF of Δ in the interval $0 \leq \Delta < \sigma$, can be computed using a renewal process analysis, e.g., as discussed in Ref. 17.

Taking the expectation of P_Δ (noting that it is a function of the random duration Δ) yields

$$\bar{P}_\Delta = [1 - F_\delta(\sigma)]P_\sigma[\theta_0, b_i(\theta_0)] + \int_0^\sigma P_\beta[\theta_0, b_i(\theta_0)]p_\Delta(\beta)\,d\beta \tag{B4}$$

A similar expression can be derived for $\bar{Q}_\Delta$.

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